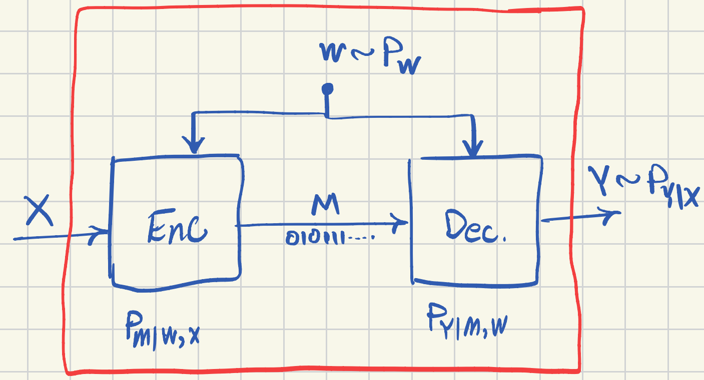
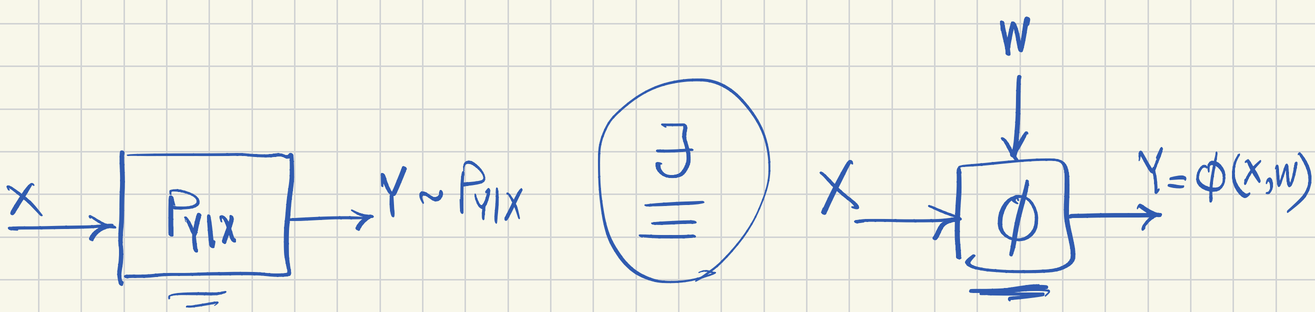
$L^* := \inf \ E[|M|]$ over all imaginable CS schemes?



Channel Simulation (CS)

Functional Representation Lemma (FRL).



- $X \perp\!\!\!\perp W$
- $Y \sim P_{Y|X}$

Proof. $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ X, Y discrete

$\mathcal{P} := \{ F_{Y|X}(y|x) : (x,y) \in \mathcal{X} \times \mathcal{Y} \}$

$W \perp\!\!\!\perp X$ and $F_W(w) \in \mathcal{P}$

$$\phi(x, w) := \min\{ y' \in \mathcal{Y} : F_W(w) \leq F_{Y|X}(y'|x) \}$$

WTS: $F_{Y|X}(y|x) = \underline{P\{ \phi(X,W) \leq y \mid X = x \}}$

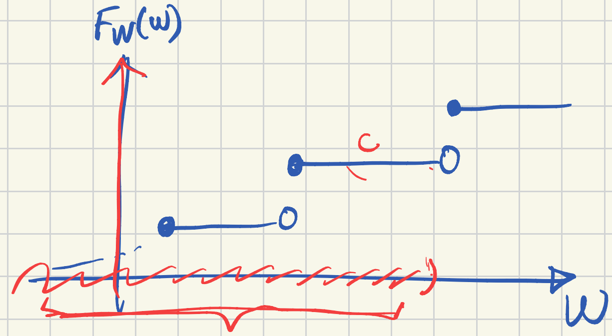
$$P\{ \phi(X,W) \leq y \mid X = x \}$$

$$= P\{ \underbrace{\phi(x, W) \leq y} \mid X = x \}$$

lemma: $\{ \phi(x,W) \leq y \} \iff \{ F_W(W) \leq F_{Y|X}(y|x) \}$

$$= P\{ F_W(W) \leq F_{Y|X}(y|x) \mid X = x \}$$

$$= P\{ F_W(W) \leq \underbrace{F_{Y|X}(y|x)}_{\text{const}} \}$$



$$= P\{ w \in \mathcal{W} : \underline{F_W(w) \leq F_{Y|X}(y|x)} \}$$

c

$$= c = F_{Y|X}(y|x)$$

$\blacksquare$

FRL and CS interplay

$\boxed{\Leftarrow}$ deteministic CS as FRL

$$\left[\begin{array}{l} W \sim P_W \\ \{C_w\}_{w \in \mathcal{W}} \\ f: \mathcal{W} \times \mathcal{X} \to \{0,1\}^* \\ g: \mathcal{W} \times \{0,1\}^* \to \mathcal{Y} \end{array}\right] \Longrightarrow \begin{array}{l} W \sim P_W \\ \phi(x,w) = g(w, f(x,w)) \end{array}$$

CS

FRL

[3] If encoding and decoding are stochastic, we can always move the local randomness at the encoder and the decoder to the unlimited common randomness W. Therefore, without loss of generality, we can assume that encoding and decoding are deterministic, and all the randomness in the scheme is contained in W. This argument works only when privacy is not a concern, since moving the local randomness at the encoder to W may harm the privacy, considering that the decoder has access to W.

$\boxed{\Rightarrow}$ FRL as CS

$$\left[\begin{array}{l} W \sim P_W \\ \underline{\phi}: \mathcal{W} \times \mathcal{X} \to \mathcal{Y} \end{array}\right] \equiv \begin{array}{l} W \sim P_W \\ M = \mathrm{HuffEnc}(\phi(x,w)) \\ Y = \mathrm{HuffDec}(M) \end{array}$$

FRL

CS

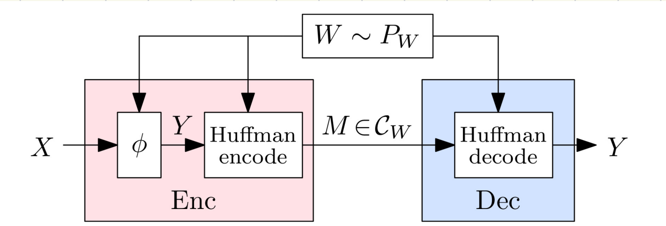


Figure 3.2: Functional representation and channel simulation.

Lossless Compression using Huffman; bounds.

$$\boxed{H(Y|W) \leq E[|M|] \leq H(Y|W) + 1}$$

Use FRL as proxy to optimal CS:

Minimal Conditional Entropy

$$H^* := \inf_{P_{W|X,Y}:\ W \perp\!\!\!\perp X,\ H(Y|X,W)=0} H(Y|W)$$

$\propto$ CS problem translates into designing $P_{Y|W}(y|w)$. (code construction for CS)

$$H^* \leq L^* \leq H^* + 1$$

inf H(Y|W)

Prop 5

$$I(X;Y) \leq H^* \leq L^* \leq H^* + 1$$

WTS: $H(Y|W) \geq I(X;Y)$

let $W \perp\!\!\!\perp X$, $\boxed{H(Y|X,W)=0}$:

$$H(Y|W) = I(X;Y|W) + \cancel{H(Y|X,W)}$$

≥ 0

$$= I(X;Y|W) + \underbrace{I(X;W)}_{=0}$$

$$= I(X;Y,W)$$

$$= \underbrace{I(X;W|Y)}_{\geq 0} + I(X;Y)$$

$$\underbrace{\inf H(Y|W)}_{H^*} \geq \underbrace{I(X;Y)}_{\alpha} \qquad \blacksquare$$