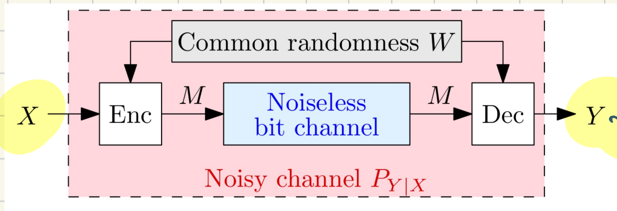
3.2 Rejection Sampling

Review:

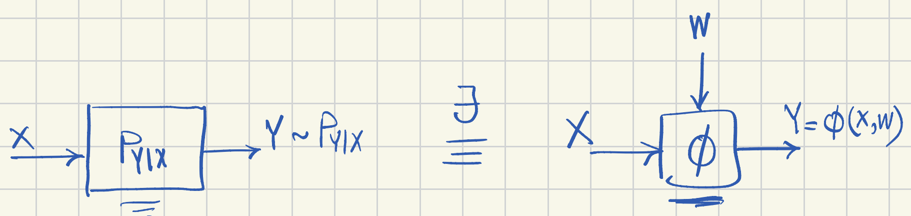
⓪ Channel simulation:



$L := E[|M|]$

goal: $L^* = \inf L$

①



- $X \perp\!\!\!\perp W$
- $Y \sim P_{Y|X}$

② Using Functional Representation Lemma in Channel Simulation setup:

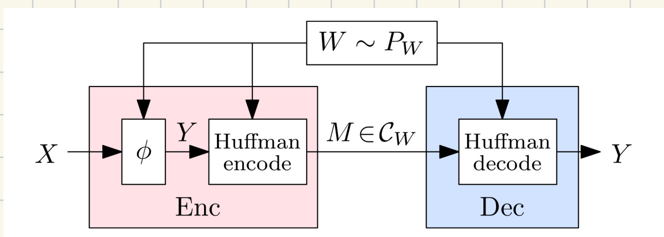


Figure 3.2: Functional representation and channel simulation.

$L := E[|M|]$

goal: $L^* := \inf E[|M|]$

Theorem 4:

$$I(X;Y) \leq L^* \leq I(X;Y) + \log_2(I(X;Y)+2) + 3$$

③

$$H(Y|W) \leq E\big[|M|\big] \leq H(Y|W) + 1$$

let $H^* := \inf H(Y|W)$ then

$$I(X;Y) \leq H^* \leq L^* \leq H^* + 1$$

Final Results

3.2.1 Classical Rejection Sampling: $H(Y|W) \leqslant -\log_2 \gamma - \frac{1-\gamma}{\gamma} \log_2 (1-\gamma)$

3.2.2 Greedy Rejection Sampling: $H(Y|W) \leqslant I(X;Y) + \log_2 \left(I(X;Y) + \log_2 4e \right) + \log_2 4e$

Classical Rejection Sampling

Goal: sample $Y \sim P$

Given: $\bar{Y}_1, \bar{Y}_2, \ldots \overset{iid}{\sim} Q$ (Knowing that $P \ll Q$)

$U_1, U_2, \ldots \overset{iid}{\sim} U_{[0,1]}$

global scale $\gamma \in (0,1)$ such that: $\gamma \frac{dP}{dQ}(y) \leqslant 1 \quad \forall y \in \mathcal{Y}$

(acceptance probability)

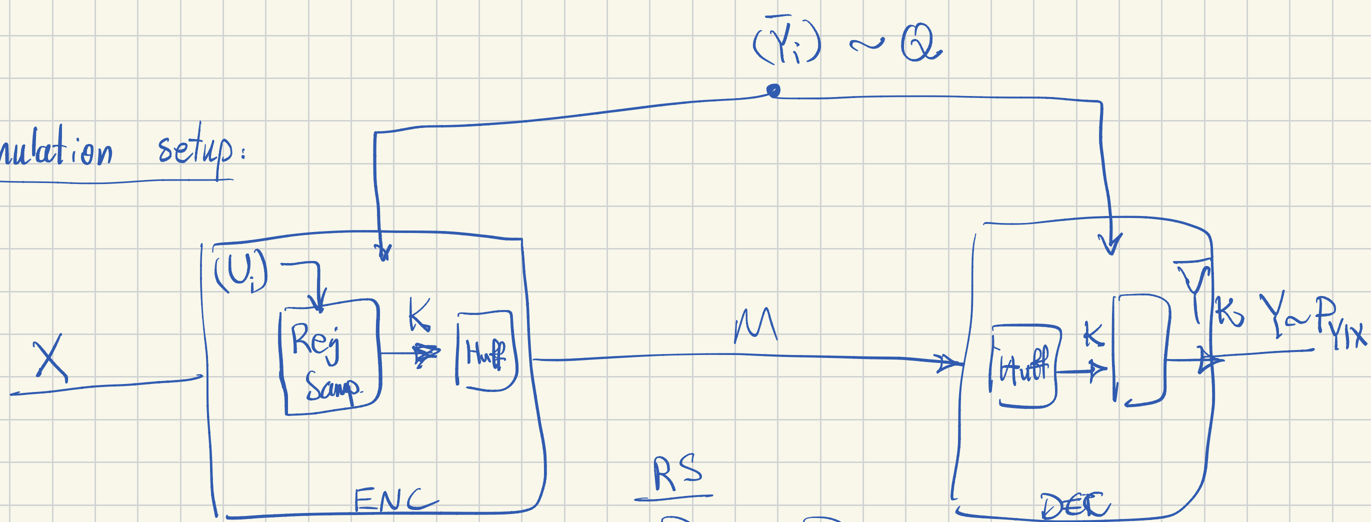
discrete: $\gamma \frac{P(y)}{Q(y)} \leqslant 1$

Rule: accept variate $\bar{Y}_i$ as desired Y if $\boxed{U_i \leqslant \gamma \frac{P(\bar{Y}_i)}{Q(\bar{Y}_i)}}$

$$Y := \bar{Y}_K \quad \text{with} \quad K := \min \left\{ i \in \mathbb{N} : U_i \leqslant \gamma \frac{P(\bar{Y}_i)}{Q(\bar{Y}_i)} \right\}$$

one can prove that $Y \sim P$. (look at Karl Sigman's note on "Acceptance-Rejection Method")

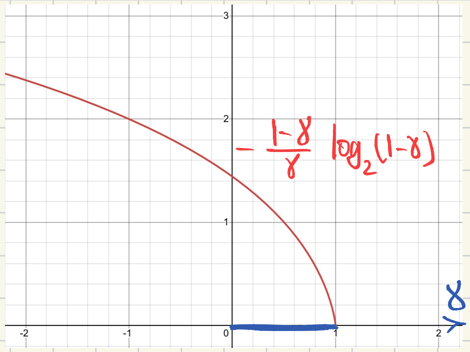
Channel simulation setup:



$P := P_{Y|X}$

Thm 6. In RS setting we have,

$$H(Y|W) \leq -\log_2 \gamma - \frac{1-\gamma}{\gamma} \log_2 (1-\gamma)$$



Claim generally $I(X;Y) \leq -\log_2 \gamma$.

Loose bound, why? γ fixed at all iterations.

Greedy Rejection Sampling

Rule: accept $\bar{Y}_i$ as Y if : $\boxed{U_i \leq \rho_i(\bar{Y}_i)}$

iteration index i

$$Y = \bar{Y}_K, \quad \text{with} \quad K := \min\{ i \in \mathbb{N} : U_i \leq \rho_i(\bar{Y}_i)\}$$

Task:

Design acceptance probabilities at each iteration: $\rho_i(y) \in [0,1]$
$y \in \mathcal{Y}$

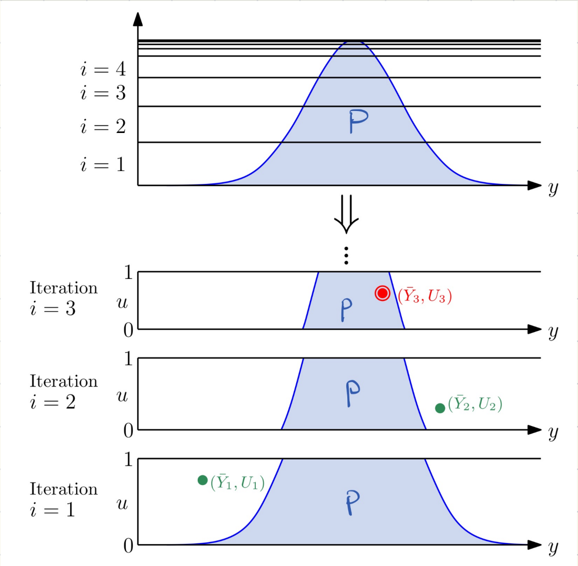
Greedily maximize chance of acceptance at iteration i.

$\vdots$

$$\rho_k(y) = \min\left\{ \frac{1}{S_k}\left(\frac{dP}{dQ}(y) - \sum_{j=1}^{k-1} \rho_j(y) S_j\right), 1\right\},$$

$$S_k = S_{k-1} \cdot \left(1 - E[\rho_{k-1}(\bar{Y})]\right),$$

with $S_k := \mathbb{P}(K \geq k)$.



(Theorem 7)

↓

Corollary 8.

Channel simulation setting under Greedy RS:

$$H(Y|W) \leq \underset{\sim\sim\sim\sim\sim}{I(X;Y)} + \log_2\left(I(X;Y) + \log_2 4e\right) + \log_2 4e$$